# Vacuum fluctuations and Casimir force

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First lecture :

a short history of quantum fluctuations of the electromagnetic field - from Planck and Einstein to Casimir

a brief introduction to modern quantum optics with a few examples of applications to quantum noise reduction

Second and third lectures :

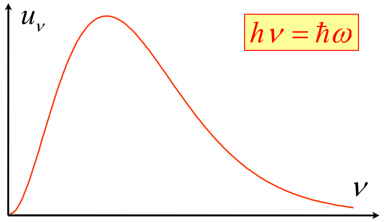
the Casimir force between mirrors in vacuum

## A short history of quantum fluctuations…

- Thermal fluctuations of electromagnetic field (EMF) in an enclosure at thermal equilibrium (Planck, 1900)
  - mean energy per mode

$$\overline{E} = \overline{n} h\nu = \overline{n}\hbar\omega$$

$$\overline{n} = \frac{1}{e^{\hbar\omega/k_B T} - 1}$$




$h\nu = \hbar\omega$

- This solves the puzzle of blackbody radiation
  - mean energy density (energy per unit volume)

$$u = \sum_{\text{modes}} \overline{n}\hbar\omega = \int u_\nu d\nu = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3}$$

Stefan-Boltzmann law

## Second Planck law (1912)

- Zero-point fluctuations (ZPF) still exist at the limit of zero temperature
  - mean energy per mode

$$\overline{E} = \overline{n}\hbar\omega + \frac{\hbar\omega}{2} \qquad \overline{n} = \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

  - ZP motion initially attributed to matter only

- ZPF have physical effects :
  - The intensity of Bragg scattering is reduced, due to motion of atoms about their equilibrium positions; zero-point motion contributes to the reduction (Debye, 1914)
  - First experimental observation through a comparison of vibrational spectra of $B^{10}O^{16}$ and $B^{11}O^{16}$ (Mulliken, 1924)

*P.W. Milonni & M.L. Shih, Am. J. Phys. 59 684 (1990)*

## Low and high-temperature limits

- Fluctuations still present at zero temperature
  - mean energy per mode

$$E = \frac{\hbar\omega}{2} \quad , \quad T = 0$$

- High-temperature limit leads to the first correct demonstration (Einstein & Stern, 1913)

$$\overline{n}\hbar\omega = k_B T - \frac{\hbar\omega}{2} + O\left(\frac{1}{T}\right) \quad , \quad T \to \infty$$

$$\overline{n}\hbar\omega + \frac{\hbar\omega}{2} = k_B T + O\left(\frac{1}{T}\right) \quad , \quad T \to \infty$$

ZPF needed to reproduce the expected classical limit

## Nernst (1916)

- Birth of vacuum fluctuations = ZPF of the EMF

- Vacuum energy (calculated by integrating over the modes) is much larger than the energy observed (through gravitational phenomena) in free space
  - mean energy density (energy per unit volume)

$$u = \sum_{\text{modes}} \left(\overline{n} + \frac{1}{2}\right)\hbar\omega = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3} + \frac{1}{8\pi^2}\frac{(\hbar\omega_{\max})^4}{(\hbar c)^3}$$

$\omega_{\max}$ cutoff frequency

- This “vacuum energy puzzle” was first discussed by Nernst in 1916, it is still unsolved today

*Reynaud et al in “On the nature of dark energy” (2002) quant-ph/0210173*

## The “vacuum catastrophe”

From a conservative estimation …

Solar system / QCD energy

$$\frac{\rho^{\text{observ}}}{\rho^{\text{calcul}}} \approx 10^{-40}$$

… to the “largest discrepancy ever observed” between theory and experiment !

Cosmology / Planck energy

$$\frac{\rho^{\text{observ}}}{\rho^{\text{calcul}}} \approx 10^{-120}$$

*R.J. Adler et al, Am. J. Phys. (1995)*

## An extreme position

« For fields [in contrast to the material oscillator], it is more consistent not to introduce the zero-point energy »

« for, on the one hand, the latter would give rise to an infinitely large energy per unit volume due to the infinite number of degrees of freedom... and, as is evident from experience, it does not produce any gravitational field... »

Ok !

« and, on the other hand, it would be unobservable since it cannot be emitted, absorbed or scattered, and hence, cannot be contained within walls. »

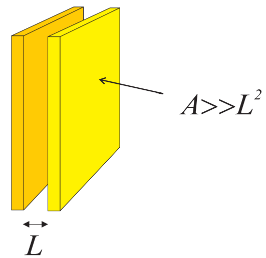
No !

W. Pauli *Wave mechanics* (1933)

## Vacuum fluctuations are predicted by quantum theory

- They have well-known physical effects from the microscopic scale
  - Spontaneous emission of atoms
  - Van der Waals forces, Lamb shift in atoms …
  - Radiative corrections in subatomic physics
- to the macroscopic scale
  - Casimir forces (1949)

$$F_{\text{Cas}} = -\frac{\hbar c \pi^2}{240\, L^4} A$$

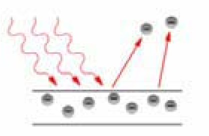



… and they are directly studied in quantum optics

## Einstein and the quanta of light

- 1905 :
  - Light (really) constituted of quanta (to be called “photons” after 1921)
  - Each light-quantum has an energy $E = \hbar\omega$
  - Law of the photoelectric effect

$$\boxed{K_e = \hbar\omega - W > 0}$$



- Also in 1905 :
  - Frequency and energy of a light-quantum modified analogously under a Lorentz transformation ( *ie number of quanta is a Lorentz invariant* )
  - Brownian motion explained in terms of atomic fluctuations

## Einstein and fluctuations of light

- 1909 :
  - Number of quanta contained in a given region of space shows fluctuations

$$\boxed{\delta n = n - \overline{n}}$$

  - These fluctuations can be characterized quantitatively by the variance

$$\boxed{\Delta n^2 = \overline{(\delta n)^2} = \overline{n^2} - \overline{n}^2}$$

- First quantitative discussion of “wave-particle duality” :
  - This variance is the sum of two terms corresponding respectively to a wave and particle interpretation

$$\boxed{\Delta n^2 = \overline{n}^2 + \overline{n}}$$

*Reynaud in Pour La Science Spécial Einstein (Décembre 2004)*

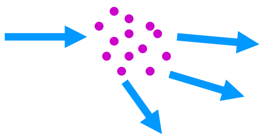
## Einstein and random processes

- 1917 :
  - Each light-quantum not only has an energy but also a momentum
  - Energy and momentum are conserved in each individual emission or absorption process

$$\boxed{E = \hbar\omega} \qquad \boxed{\vec{p} = \hbar\vec{k}}$$

- Compton effect (1922)

$$\boxed{\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)}$$



- Also in the paper of 1917 :
  - Spontaneous processes (independent of the number of incident photons) superimposed to stimulated processes (triggered by incident photons)

## Einstein and statistics of “bosons”

- 1924 : Bose gives a new derivation of Planck law
- 1924-1925 : Einstein supports the publication of Bose results and extends this new derivation
  - invents the concept of “boson” by generalizing Bose method to matter particles,
  - discovers “Bose-Einstein condensation”,
  - computes the fluctuations of the number of particles in a region of space …
    - The variance is anew the sum of two terms corresponding to wave and particle contributions
    - The wave term is associated to de Broglie waves
    - This remark leads Schrödinger to the wave equation

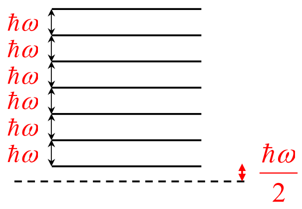
*A. Pais Rev. Mod. Phys. 21 (1949)*

## Introduction to quantum optics of one mode

- One mode of the free electromagnetic field
  = an harmonic oscillator (angular frequency ω)
  - quantized energy levels

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

  - the state $|n\rangle$ contains $n$ photons




- The vacuum state simply corresponds to the lowest energy state $n = 0$
  - the number of photons is zero in vacuum BUT
  - the field does not vanish
  - the energy is not zero $E_0 = \frac{\hbar\omega}{2}$

## Quantized photon number

- Field in a given mode, at a given position,

$$\mathcal{E}(t) = \mathcal{E}_0\left(ae^{-i\omega t} + a^+e^{i\omega t}\right) = \mathcal{E}_0\left(a_1 \cos\omega t + a_2 \sin\omega t\right)$$

$$a_1 = a + a^+ \quad , \quad a_2 = -i\left(a - a^+\right)$$

- Field components do not commute

$$\left[a, a^+\right] = aa^+ - a^+a = 1 \quad , \quad \left[a_1, a_2\right] = a_1a_2 - a_2a_1 = 2i$$

  - Heisenberg inequality $\Delta a_1 \times \Delta a_2 \geq 1$

- Energy is a quadratic form of the fields

$$E = \hbar\omega\frac{a^+a + aa^+}{2} = \hbar\omega\left(n + \frac{1}{2}\right)$$

photon number

$$n = a^+a$$

## Field covariance matrix

- Field fluctuations

$$\delta a_1 = a_1 - \overline{a_1} \quad , \quad \delta a_2 = a_2 - \overline{a_2}$$

  may be represented by a covariance matrix

$$\Delta = \begin{bmatrix} \overline{(\delta a_1)^2} & \overline{\delta a_1 \delta a_2} \\ \overline{\delta a_2 \delta a_1} & \overline{(\delta a_2)^2} \end{bmatrix} = \begin{bmatrix} \Delta a_1^{\,2} & \overline{\delta a_1 \cdot \delta a_2} + i \\ \overline{\delta a_1 \cdot \delta a_2} - i & \Delta a_2^{\,2} \end{bmatrix}$$

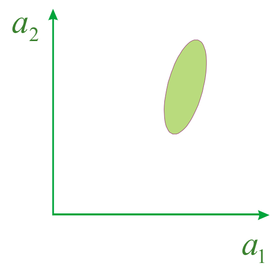
$$\Delta a_1^{\,2} = \overline{(\delta a_1)^2} \quad , \quad \Delta a_2^{\,2} = \overline{(\delta a_2)^2} \quad , \quad \overline{\delta a_1 \cdot \delta a_2} = \frac{1}{2}\overline{\delta a_1 \delta a_2 + \delta a_2 \delta a_1}$$

- Positivity of the matrix Δ leads to a generalized Heisenberg inequality

$$\Delta a_1^{\,2} \times \Delta a_2^{\,2} - \left(\overline{\delta a_1 \cdot \delta a_2}\right)^2 \geq 1$$

## Representation in phase space

- Equivalent graphical representation in phase space




  - A quantum state is not represented by a point but by a distribution in phase space
  - Covariances described by the geometry of the elliptical distribution

- The generalized Heisenberg inequality means that the area of the distribution in phase space is larger than a minimal value $\pi$

*S. Reynaud, Annales de Physique (1990)*

## Minimum uncertainty states

- Quantum states having the closest vicinity to classical fields, defined by :
  - minimal area of the distribution in phase space

$$\Delta a_1^{\,2} \times \Delta a_2^{\,2} - \left(\overline{\delta a_1 \cdot \delta a_2}\right)^2 = 1$$

- This condition is sufficient for defining minimal states as “nearly classical” :
  - eigenstates of generalized annihilation operators
  - gaussian wavefunctions in $a_1 -$ or $a_2 -$ representations
  - gaussian Wigner distributions

$$A\left|\mathrm{MUS}\right\rangle = \overline{A}\left|\mathrm{MUS}\right\rangle$$

$$A = a\cosh\xi - a^{+} e^{2i\varphi}\sinh\xi$$

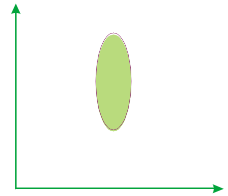
$\xi \Leftrightarrow$ squeezing of the ellipse

$\varphi \Leftrightarrow$ direction of the axis

## Particular minimal states

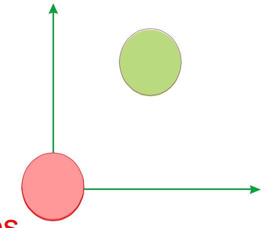
- Uncorrelated minimal states :
  - also called squeezed states

$$\overline{\delta a_1 \cdot \delta a_2} = 0 \quad \rightarrow \quad \Delta a_1 \times \Delta a_2 = 1$$



- Phase insensitive minimal states :
  - also called coherent states

$$\overline{\delta a_1 \cdot \delta a_2} = 0 \quad , \quad \Delta a_1 = \Delta a_2 = 1$$

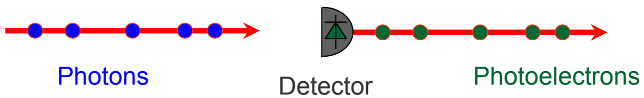


- Vacuum fluctuations
  = fluctuations of the coherent states
  - vacuum may be defined as the coherent state centered on a null mean field

## Photon noise in a light beam

- In a (too simple) corpuscular interpretation :
  - the photon noise is characterized by a series of times representing the “positions” of photons or their detection events

$$I(t) = \sum_k \delta(t - t_k)$$




- Photon statistics is characterized by the correlation function or, equivalently, by the noise spectrum

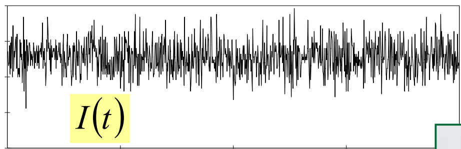
$$C_I(\tau) = \overline{I(t)I(t+\tau)} - \overline{I(t)}\;\overline{I(t+\tau)}$$

$$S_I(\omega) = \int d\tau\, e^{i\omega\tau}\, C_I(\tau)$$

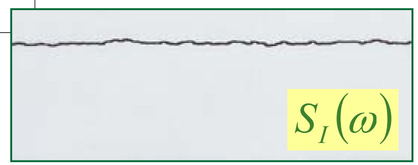
*S. Reynaud, Annales de Physique (1990)*

## Poisson statistics

- The simplest random point process :
  - probability of next photon independent of the history
  - successive delays are not correlated
  - exponential distribution for the delays
  - theoretically the output of an ideal laser
- Easy numerical simulation



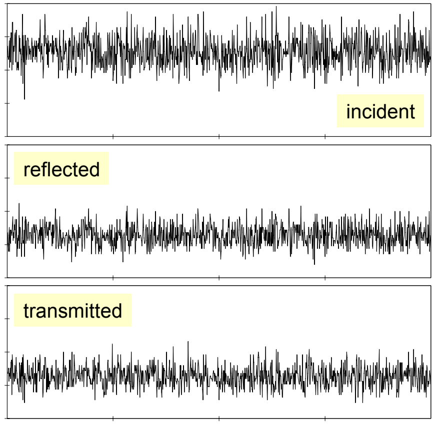

Numerical generation of uncorrelated delays with an exponential distribution

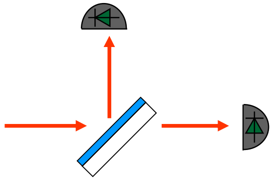



The noise spectrum is flat

## Poisson Statistics and beam splitting

- The simplest random choice process :
  - probability of reflection/transmission not correlated for different photons



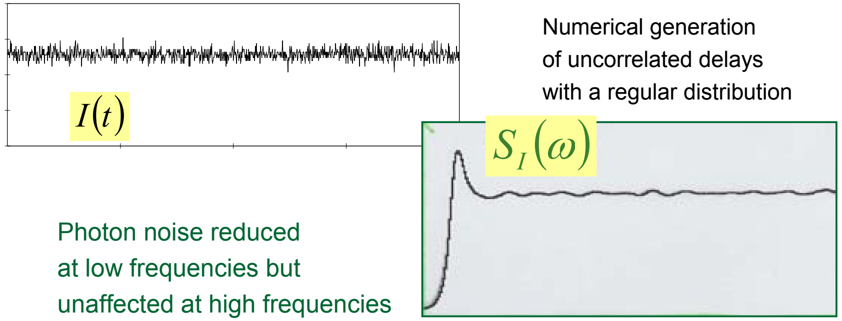
Numerical generation
of a Poisson statistics
+ 50/50 splitting process

Flat noise spectrum
for the three noises

## Sub-Poisson statistics

- A simple example of sub-Poisson statistics :
  - decimation (ie, selection of the next tenth photon) in a Poisson statistics with ten times more photons
  - successive delays are still uncorrelated
  - but the distribution of delays is more regular

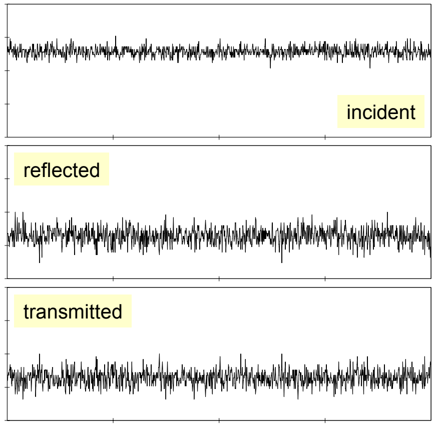
Numerical generation
of uncorrelated delays
with a regular distribution

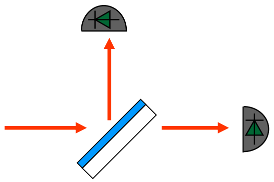



Photon noise reduced
at low frequencies but
unaffected at high frequencies

## Sub-Poisson statistics and beam splitting

- Same random splitting process :
  - probability of reflection/transmission not correlated for different photons



Numerical generation
of sub-Poisson statistics
+ 50/50 splitting process

Reduction of the photon noise
degraded by the random splitting

## Quantum optics of the light beam

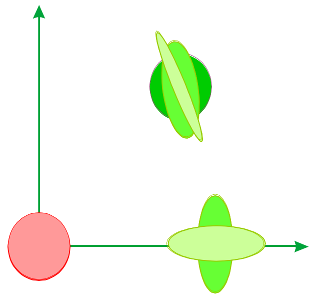
- Electromagnetic field in a light beam
  = an infinity of field modes
  = an infinity of harmonic oscillators
  - this multiplicity of modes is needed to reproduce the dynamics of fluctuations in space and time
- Simplest model
  - one-dimensional space (2-d spacetime) and scalar field theory (transverse modes and polarizations ignored)
  - two propagation directions $\Phi(t,x) = \varphi^+(t-x) + \varphi^-(t+x)$
  - each with a mode decomposition

$$\varphi(t) = \int_0^\infty \frac{d\omega}{2\pi}\sqrt{\frac{\hbar}{2\omega}}\left(a_\omega e^{-i\omega t} + a_\omega{}^+ e^{i\omega t}\right)$$

$$a_\omega a_{\omega'}^+ - a_{\omega'}^+ a_\omega = 2\pi\delta(\omega-\omega')$$

## Representation in phase space

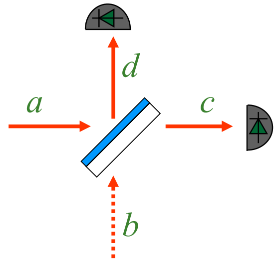
- Fluctuations depend on frequency :
  - a covariance matrix for each frequency
- Fluctuations can be squeezed :
  - in amplitude : along the mean field
  - in phase : in quadrature with the mean field
  - …



- Vacuum state defined with a null mean field and fluctuations of coherent states at all frequencies

## Quantum theory of the beam splitting

- A beam splitter has two output but also two input ports :
  - the usually ignored second input port plays a key role in the process
  - the output fields are




$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} t & ir \\ ir & t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

unitary S-matrix

$$r^2 + t^2 = 1$$

- the intensities are deduced

$$c^+c = t^2a^+a + irt\left(a^+b - b^+a\right) + r^2b^+b$$

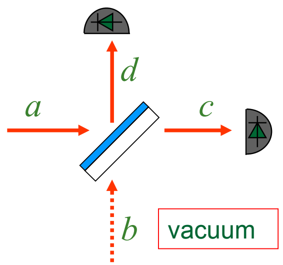
$$d^+d = r^2a^+a - irt\left(a^+b - b^+a\right) + t^2b^+b$$

total number of photons preserved

$$c^+c + d^+d = a^+a + b^+b$$

## Fluctuations in the beam splitting

- We first assume that vacuum fluctuations enter the “unused” input port




$$b^+b = 0$$

$$c^+c = t^2a^+a + irt\left(a^+b - b^+a\right)$$

$$d^+d = r^2a^+a - irt\left(a^+b - b^+a\right)$$

- in a linear treatment with $\overline{a_1} \neq 0 \quad , \quad \overline{a_2} = \overline{b_1} = \overline{b_2} = 0$
- we obtain

$$\delta\left(c^+c\right) = 2t^2\overline{a_1}\delta a_1 + 2rt\overline{a_1}\delta b_2$$

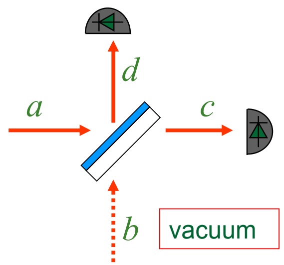
$$\delta\left(d^+d\right) = 2r^2\overline{a_1}\delta a_1 - 2rt\overline{a_1}\delta b_2$$

fluctuations in the splitting process reveal fluctuations of vacuum fields entering the “unused” port

## A consequence : loss and noise

- The beam splitter is a simple model of loss on a beam

$$\left\langle c^+c\right\rangle = \left\langle a^+a\right\rangle - \left\langle d^+d\right\rangle = \left(1 - r^2\right)\left\langle a^+a\right\rangle$$

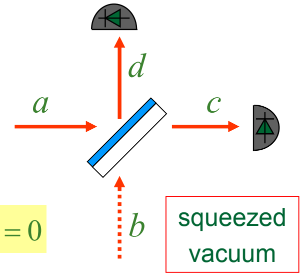



- an “excess noise” is necessarily associated to this loss mechanism
- it is exactly the noise added by the field in the “unused” input port
- even for a perfectly regular beam entering the port a, there is noise in the output port c (or d)

$$\delta\left(a^+a\right) = 0 \qquad \delta\left(c^+c\right) = 2rt\overline{a_1}\delta b_2 \qquad \delta\left(d^+d\right) = -2rt\overline{a_1}\delta b_2$$

# Applications : reduction of quantum noise

- The noise in the beam splitter can be reduced by entering squeezed fluctuations in the usually ignored second input !
  - we consider a 50/50 beam splitter $t = r = \dfrac{1}{\sqrt{2}}$
  - and assume $\overline{a_1} \neq 0 \quad , \quad \overline{a_2} = \overline{b_1} = \overline{b_2} = 0$
  - we deduce (in a linear treatment)




$$\langle c^+ c \rangle = \langle d^+ d \rangle = \frac{1}{2} \langle a^+ a \rangle \qquad \delta(d^+ d) = -\delta(c^+ c) = \frac{1}{2} \overline{a_1} \, \delta b_2$$

if vacuum fluctuations entering the "unused" port are squeezed (along $b_2$)
then the photon noise is reduced at the output of the beam splitter

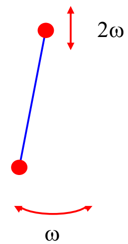
# Squeezing by parametric coupling

- Simple example
  - pendulum with a resonant frequency ω excited at frequency 2ω
  - one of the quadrature component is amplified

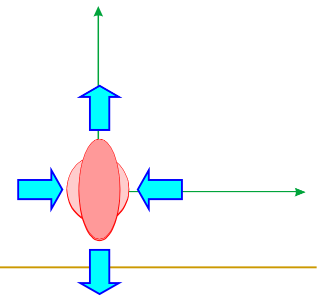
$$a_1 \quad \rightarrow \quad a_1 \, e^{\mu t}$$

  - and the other one de-amplified

$$a_2 \quad \rightarrow \quad a_2 \, e^{-\mu t}$$



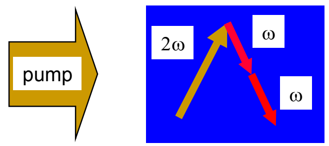

Parametric "amplification" produces a squeezing of vacuum fluctuations

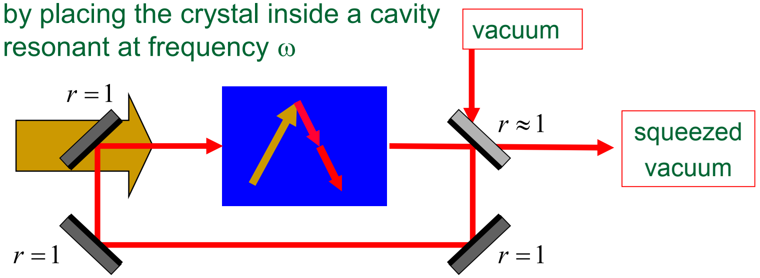


*S. Reynaud, Annales de Physique (1990)*

# Optical parametric amplifier

- Optical analog
  - "parametric" crystal excited by a pump at frequency 2ω and producing two photons at frequency ω
  - the process is resonantly enhanced by placing the crystal inside a cavity resonant at frequency ω




the whole setup is equivalent to a parametrically excited pendulum

# Application in interferometric detection

- Proposals for interferometric detection of gravitational waves have triggered the studies of quantum optical noise
  - an interferometer is a beam splitter whose output is "tuned" by the signal
  - due to the weakness of the signal, photon noise is an issue
  - photon noise can be reduced at the output of the interferometer by entering squeezed fluctuations in the "unused" input port

http://www.virgo.infn.it/

*See for example Chelkowski et al, Phys. Rev. A71 013806 (2005)*